Physical limits on the notion of very low temperatures

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Standard statistical thermodynamic views of temperature fluctuations predict a magnitude $(\sqrt{\langle(\Delta T)^2\rangle}/T) \approx \sqrt{(k_B/C)}$ for a system with heat capacity C. The extent to which low temperatures can be well defined is discussed for those systems that obey the thermodynamic third law in the form $\lim_{(T\to 0)} C=0$. Physical limits on the notion of very low temperatures are exhibited for simple systems. Application of these concepts to bound Bose condensed systems are explored, and the notion of bound boson superfluidity is discussed in terms of the thermodynamic moment of inertia.
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I. INTRODUCTION

A problem of considerable importance in low temperature physics concerns physical limitations on how small a temperature can be well defined in the laboratory [1]. In what follows, we shall consider temperature fluctuations that define a system temperature ''uncertainty''

$$\delta T=\sqrt{\langle (T_s-\langle T_s\rangle)\rangle^2}, \tag{1.1}$$

whenever a finite system at temperature T_s is in thermal contact with a large reservoir at bath temperature T. Only on the thermodynamic average do we expect the system temperature to be equal to the bath temperature; i.e., $\langle T_s\rangle \approx T$. The fluctuation from this average result [2] has the magnitude

$$\left(\frac{\delta T}{T}\right)\approx\sqrt{\frac{k_B}{C}}, \tag{1.2}$$

where C is the heat capacity of the finite system, and k_B is Boltzmann's constant. Since C is an extensive thermodynamic quantity, one expects the usual small fluctuation in temperature $(\delta T/T)\propto(1/\sqrt{N})$ in the thermodynamic limit $N\to\infty$, where N is the number of microscopic particles. However, in low temperature physics (for systems with finite values for N), temperature fluctuations are by no means required to be negligible.

For those finite sized systems that obey the thermodynamic third law

$$\lim_{T\to 0} C=0, \tag{1.3}$$

one finds from Eqs. (1.2) and (1.3) that

$$\left(\frac{\delta T}{T}\right)\to\infty \text{ as } T\to 0 \text{ with } N<\infty. \tag{1.4}$$

Equation (1.4) sets the limits on what can be regarded as the ultimate lowest temperatures for finite thermodynamic systems; i.e., the temperature must at least obey $\delta T \ll T$.

In Sec. II, the theoretical foundations for Eq. (1.4) will be discussed. In brief, the microcanonical entropy of a thermodynamic system is given by

$$S(E)=k_B \ln \Gamma(E), \tag{1.5}$$

where $\Gamma(E)$ is the number of system quantum states with energy E. The microcanonical entropy defines the *system temperature* T_s via

$$\left(\frac{1}{T_s}\right)=\left(\frac{dS}{dE}\right). \tag{1.6}$$

The *thermal bath temperature* T, which is not quite the system temperature T_s, enters into the canonical free energy

$$F(T)=-k_BT \ln Z(T), \tag{1.7}$$

where the partition function is defined as

$$Z(T)=Tr(e^{-H/k_BT})=\sum_E \Gamma(E)e^{-E/k_BT}. \tag{1.8}$$

The probability distribution for the energy of the system, when in contact with a thermal bath at temperature T, is given by

$$P(E;T)=\left(\frac{\Gamma(E)}{Z(T)}\right)\exp\left(\frac{-E}{k_BT}\right)=\exp\left(\frac{F(T)-E+TS(E)}{k_BT}\right), \tag{1.9}$$

as dictated by Gibbs. Thus, the temperature T (of the thermal bath) does not fluctuate while system energy E does fluctuate according to the probability rule of Eq. (1.9). On the other hand, the system temperature $T_s(E)$ in Eq. (1.6) depends on the system energy and thereby fluctuates since E fluctuates. It is only for the energy E^* of maximum probability $\mathrm{Max}_E P(E;T)=P_{\max}=P(E^*;T)$ that the system temperature is equal to the bath temperature $T_s(E^*)=T$. If the energy probability distribution is spread out at low thermal bath temperatures, then temperature fluctuations are well defined in the canonical ensemble of Gibbs.

In Sec. III, temperature fluctuations are illustrated using the example of blackbody radiation in a cavity of volume V. For this case, it turns out that the thermal wavelength Λ_T of the radiation in the cavity,

$$\Lambda_T = \left(\frac{\hbar c}{k_B T}\right), \tag{1.10}$$

must be small on the scale of the cavity length $V^{1/3}$; i.e., $\Lambda_T \ll V^{1/3}$. For example, in a cavity of volume $V \sim 1\ \text{cm}^3$, the lowest temperature for the radiation within the cavity is of order $T_{\min} \sim 1$ K. It is of course possible to cool the conducting metal walls of a cavity with a length scale ~ 1 cm to well below 1 K. However, this by no means implies that the radiation within the cavity can have a temperature well below 1 K. The point is that at temperatures $T_s < 1$ K, there are perhaps only a few photons (in total) in the cavity. The total number of photons are far too few for the cavity radiation system temperature to be well defined.

In Sec. IV, a confined system of $\mathcal{N}$ atoms obeying ideal gas Bose statistics is discussed. Such systems can be Bose condensed, and are presently (perhaps) the lowest temperature systems available in laboratories. In the quasiclassical approximation, the free energy is computed in Sec. V. Questions concerning bounds on ultralow temperatures are explored. Whether or not such Bose condensed atoms can exhibit superfluid behavior is discussed in Sec. VI. The superfluid and normal fluid contributions to the moment of inertia are computed. In the concluding Sec. VII, another simple system with fluctuation limits on ultralow temperatures will be briefly discussed.

II. THEORETICAL FOUNDATIONS

Let $\phi(E)$ denote some physical quantity that depends on the energy E of a physical system. If the system is in contact with a thermal bath at temperature T, then the thermodynamic average may be calculated from

$$\langle \phi \rangle = \sum_E P(E;T)\phi(E), \tag{2.1}$$

where the probability $P(E;T)$ has been defined in Eq. (1.9). Using the "summation by parts" [3,4] formula

$$\sum_E \frac{\partial}{\partial E}[P(E;T)\phi(E)] = 0, \tag{2.2}$$

i.e., with a strongly peaked $P(E;T)$,

$$-\sum_E P(E;T)\left(\frac{d\phi(E)}{dE}\right) = \sum_E \phi(E)\left(\frac{\partial P(E;T)}{\partial E}\right), \tag{2.3}$$

one finds

$$-k_B\left\langle \frac{d\phi(E)}{dE}\right\rangle = k_B \sum_E \phi(E)\left(\frac{\partial P(E;T)}{\partial E}\right). \tag{2.4}$$

Employing Eqs. (1.6) and (1.9),

$$k_B\left(\frac{\partial P(E;T)}{\partial E}\right) = \left(\frac{1}{T_s(E)} - \frac{1}{T}\right)P(E;T). \tag{2.5}$$

Equations (2.4) and (2.5) imply the central result of this section:

$$-k_B\left\langle \frac{d\phi}{dE}\right\rangle = \left\langle \left(\frac{1}{T_s} - \frac{1}{T}\right)\phi \right\rangle. \tag{2.6}$$

If we choose $\phi(E) = 1$, then Eq. (2.6) reads

$$\left\langle \frac{1}{T_s}\right\rangle = \frac{1}{T}; \tag{2.7}$$

i.e., on average, the reciprocal of the system temperature is equal to the reciprocal of the thermal bath temperature. Thus, with fluctuations from the mean

$$\Delta\phi = \phi - \langle \phi \rangle, \tag{2.8}$$

$$\Delta\left(\frac{1}{T}\right) = \frac{1}{T_s} - \left\langle \frac{1}{T_s}\right\rangle = \frac{1}{T_s} - \frac{1}{T}, \tag{2.9}$$

Eq. (2.6) reads

$$-k_B\left\langle \frac{d\phi}{dE}\right\rangle = \left\langle \Delta\left(\frac{1}{T}\right)\Delta\phi \right\rangle. \tag{2.10}$$

If we choose in Eq. (2.10) the function ϕ to be

$$\phi = \left(\frac{1}{T_s}\right) \quad \text{and} \quad -\left(\frac{d\phi}{dE}\right) = \frac{1}{T_s^2}\left(\frac{dT_s}{dE}\right) = \left(\frac{1}{T_s^2 C}\right) \tag{2.11}$$

[where $C = (dE/dT_s)$ is the system heat capacity], then

$$\left\langle \Delta\left(\frac{1}{T}\right)^2 \right\rangle = \left\langle \frac{1}{T_s^2}\left(\frac{k_B}{C}\right)\right\rangle. \tag{2.12}$$

The standard Eqs. (1.1) and (1.2) follow from the more precise Eqs. (2.7) and (2.12) in the limit of small temperature fluctuations; i.e.,

$$\langle (\Delta T)^2 \rangle \approx \left(\frac{k_B T^2}{C}\right) \quad \text{if} \quad \delta T = \sqrt{\langle (\Delta T)^2 \rangle} \ll T. \tag{2.13}$$

The condition $\delta T \ll T$ is required in order that the canonical thermal bath temperature be equivalent to the microcanonical system temperature. If the microcanonical and canonical temperatures are not equivalent, then the statistical thermodynamic definition of temperature would no longer be unambiguous. This raises fundamental questions as to the physical meaning of temperature. The view of this work is that in an ultralow temperature limit, whereby $\delta T \sim T$ for sufficiently small T, the whole notion of system temperature is undefined, although the notion of a thermal bath temperature retains validity.

III. BLACKBODY RADIATION EXAMPLE

The heat capacity of blackbody radiation in a cavity of volume V with the walls of the cavity at temperature T is given by [5]

$$C(\text{blackbody}) = k_B\left(\frac{4\pi^2}{15}\right)\left(\frac{V}{\Lambda_T^3}\right), \tag{3.1}$$

where Λ_T is given by Eq. (1.10). From Eqs. (1.2) and (3.1) it follows that the radiation temperature of a blackbody cavity of volume V is

$$\left(\frac{\delta T(\text{blackbody})}{T}\right) \approx \sqrt{\left(\frac{15}{4\pi^2}\right)\left(\frac{\Lambda_T^3}{V}\right)} \approx 0.6\sqrt{\left(\frac{\Lambda_T^3}{V}\right)}. \tag{3.2}$$

In order to achieve a well defined radiation temperature inside the cavity, δT(blackbody) must be small on the scale of T or equivalently $\Lambda_T \ll V^{1/3}$. As stated in Sec. I, this implies a minimum temperature of $T_{\min} \sim 1$ K for a cavity of $V \sim 1\ \text{cm}^3$.

IV. CONFINED IDEAL BOSE GAS

The grand canonical free energy of an ideal Bose gas is determined by the trace [6]

$$\Xi(T,\mu) = k_B T \ \text{tr} \ \ln(1 - e^{(\mu - h)/k_B T}), \tag{4.1}$$

where h is the one boson Hamiltonian and

$$d\Xi = -\mathcal{S} dT - \mathcal{N} d\mu \tag{4.2}$$

determines the number of bosons $\mathcal{N}$. If the one boson partition function is defined as

$$q(T) = \text{tr}(e^{-h/k_B T}), \tag{4.3}$$

then the free energy obeys

$$\Xi(T,\mu) = -k_B T \sum_{n=1}^{\infty}\left(\frac{1}{n}\right) q\left(\frac{T}{n}\right) e^{n\mu/k_B T}. \tag{4.4}$$

The mean number of bosons is

$$\mathcal{N}(T,\mu) = \sum_{n=1}^{\infty} q\left(\frac{T}{n}\right) e^{n\mu/k_B T}, \tag{4.5}$$

and the statistical entropy is given by

$$\mathcal{S}(T,\mu) = -\left(\frac{\Xi(T,\mu) + \mu\mathcal{N}(T,\mu)}{T}\right) + k_B T \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right) q'\left(\frac{T}{n}\right) e^{n\mu/k_B T}, \tag{4.6}$$

where $q'(T) = \{dq(T)/dT\}$.

Of considerable theoretical [7,8] experimental [9–11] interest is the bound boson in an anisotropic oscillator potential,

$$h = -\left(\frac{\hbar^2}{2M}\right)\nabla^2 + \left(\frac{1}{2}\right) M \mathbf{r}\cdot\hat{\omega}^2\cdot\mathbf{r} - \left(\frac{\hbar}{2}\right)\text{tr}(\hat{\omega}), \tag{4.7}$$

where

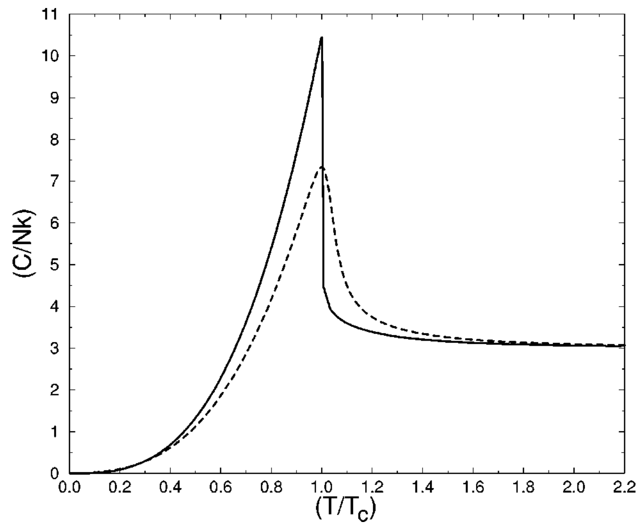

FIG. 1. The heat capacity (in units of $\mathcal{N}k_B$) is plotted as a function of temperature (in units of T_c) for $\mathcal{N} = 2\times 10^3$ atoms (dotted curve) and $\mathcal{N} = 2\times 10^6$ atoms (solid curve).

$$\hat{\omega}^2 = \begin{pmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}. \tag{4.8}$$

Equations (4.3) and (4.7) imply

$$q(T) = \prod_{j=1}^{3}\left(\frac{1}{1 - e^{-\hbar\omega_j/k_B T}}\right). \tag{4.9}$$

The heat capacity may be defined by

$$C = T\left(\frac{\partial S}{\partial T}\right)_{\mathcal{N}}, \tag{4.10}$$

which must be calculated numerically.

Shown in Fig. 1 is a plot of the heat capacity (in units of $\mathcal{N}k_B$) versus temperature (in units of the critical temperature T_c) for the case of $\mathcal{N} = 2\times 10^3$ and $\mathcal{N} = 2\times 10^6$ particles. We choose, for experimental interest [12], the frequency eigenvalues $(\omega_1/2\pi) = (\omega_2/2\pi) = 320$ Hz, and $(\omega_3/2\pi) = 18$ Hz. For finite $\mathcal{N}$, there is strictly speaking no Bose-Einstein condensation phase transition. The critical temperature T_c is therefore defined as that temperature for which the heat capacity reaches the maximum value $C_{\max} = C(T_c)$. Although phase transitions are defined in mathematics only in the thermodynamic limit $\mathcal{N} \to \infty$, for all practical purposes, a quasiclassical approximation of Eq. (4.1) in the form

$$\Xi(T,\mu) = k_B T \int\int \left(\frac{d^3r\, d^3p}{(2\pi\hbar)^3}\right) \times \ln(1 - e^{[\mu - h(\mathbf{p},\mathbf{r})]/k_B T}) \quad \text{(quasiclassical)}, \tag{4.11}$$

where

$$h(\mathbf{p},\mathbf{r}) = \frac{p^2}{2M} + \frac{1}{2} M \mathbf{r}\cdot\hat{\omega}^2\cdot\mathbf{r} \tag{4.12}$$

does yield a Bose-Einstein condensation phase transition whose heat capacity is sufficiently accurate for values of $\mathcal{N} \sim 10^5$ or higher. Thus, we regard recent experiments on Bose atoms confined in a magnetic bottle to be probing a physical Bose-Einstein ordered phase. Let us consider Eq. (4.11) in more detail.

V. QUASICLASSICAL BOSE CONDENSATION

In order to evaluate Eq. (4.10) we employ the quasiclassical form [13,14] of Eqs. (4.3) and (4.11); i.e.,

$$q(T)=\int\int\left(\frac{d^3r d^3p}{(2\pi\hbar)^3}\right)e^{-h(\mathbf{p},\mathbf{r})/k_BT}=\prod_{j=1}^{3}\left(\frac{k_BT}{\hbar\omega_j}\right)=\left(\frac{k_BT}{\hbar\overline{\omega}}\right)^3, \tag{5.1}$$

where $\overline{\omega}=(\omega_1\omega_2\omega_3)^{1/3}$. From Eqs. (4.4) and (5.1), it follows that Eq. (4.11) evaluates to

$$\Xi(T,\mu)=-\hbar\overline{\omega}\left(\frac{k_BT}{\hbar\overline{\omega}}\right)^4\sum_{n=1}^{\infty}\left(\frac{1}{n^4}\right)e^{n\mu/k_BT}. \tag{5.2}$$

From Eqs. (4.2) and (5.2), the number of particles obeys

$$\mathcal{N}(T,\mu)=\left(\frac{k_BT}{\hbar\overline{\omega}}\right)^3\sum_{n=1}^{\infty}\left(\frac{1}{n^3}\right)e^{n\mu/k_BT}\quad(\mu<0). \tag{5.3}$$

With the usual definition of the ζ function:

$$\zeta(s)=\sum_{n=1}^{\infty}\left(\frac{1}{n^s}\right),\quad \mathcal{R}e(s)>1, \tag{5.4}$$

the Bose-Einstein condensation critical temperature is

$$T_c=\left(\frac{\hbar\overline{\omega}}{k_B}\right)\left(\frac{N}{\zeta(3)}\right)^{1/3}. \tag{5.5}$$

The nonzero number of bosons (below the critical temperature) in the condensate state is given by

$$\mathcal{N}_0(T)=\mathcal{N}\left\{1-\left(\frac{T}{T_c}\right)^3\right\}\quad(T<T_c). \tag{5.6}$$

Finally, the entropy below the critical temperature,

$$\mathcal{S}(T)=4k_B\zeta(4)\left(\frac{k_BT}{\hbar\overline{\omega}}\right)^3=Nk_B\left(\frac{4\zeta(4)}{\zeta(3)}\right)\left(\frac{T}{T_c}\right)^3\quad(T<T_c), \tag{5.7}$$

obeys the thermodynamic third law $\lim_{(T\to 0)}\mathcal{S}(T)=0$. The heat capacity in the Bose-Einstein condensed phase is then given by

$$\left(\frac{C}{\mathcal{N}k_B}\right)=\left(\frac{12\zeta(4)}{\zeta(3)}\right)\left(\frac{T}{T_c}\right)^3\approx 10.81\left(\frac{T}{T_c}\right)^3\quad(T<T_c). \tag{5.8}$$

Employing Eqs. (1.2) and (5.8) we find that the temperature uncertainty below the critical temperature obeys

$$\left(\frac{\delta T}{T}\right)\approx\left(\frac{0.3}{\sqrt{\mathcal{N}}}\right)\left(\frac{T_c}{T}\right)^{3/2}\quad(T<T_c). \tag{5.9}$$

Thus, for $\mathcal{N}\sim 10^5$ one may safely consider the temperature of the ordered phase to be well defined in the range $T_c>T>T_{\min}$, where $T_{\min}\sim 0.05T_c$. The open question as to whether the ordered phase is a superfluid may now be considered.

VI. SUPERFLUID FRACTION OF THE BOUND BOSON SYSTEM

The notion of a superfluid fraction in an experimental Bose fluid (such as liquid ^{4}He) may be viewed in the following manner: Suppose that we pour the liquid into a very slowly rotating vessel and close it off from the environment. The walls of the vessel are at a bath temperature T, and the vessel itself rigidly rotates at a very small angular velocity $\mathbf{\Omega}$. In the ''two-fluid'' model [15,16], the normal part of the fluid rotates with a rigid body angular velocity $\mathbf{\Omega}$, which is the same as the angular velocity of the vessel. On the other hand, the superfluid part of the fluid does not rotate. The superfluid exhibits virtually zero angular momentum for sufficiently small $\mathbf{\Omega}$. The total fluid moment of inertia tensor $\hat{I}$ is defined by the fluid angular momentum $\mathbf{L}=\hat{I}\cdot\mathbf{\Omega}$ (as $\mathbf{\Omega}\to\mathbf{0}$). We here take the limit $\mathbf{\Omega}\to\mathbf{0}$, to avoid questions concerning the effects of vortex singularities on the superfluid. The normal fluid, which rotates along with the rotating vessel, contributes to the fluid moment of inertia. The superfluid, which does not rotate with the vessel, does not contribute to the moment of inertia. Thus, the *geometric* moment of inertia,

$$\hat{I}_{ij}^{\text{geometric}}=\int d^3r\,\overline{\rho(\mathbf{r})}(r^2\delta_{ij}-r_ir_j), \tag{6.1}$$

where $\overline{\rho(\mathbf{r})}$ is the mean mass density of the fluid (at rest), overestimates the *physical* moment of inertia eigenvalues when the fluid is actually a superfluid. The normal fluid contributes to the moment of inertia and the superfluid does not do so in the limit $\mathbf{\Omega}\to\mathbf{0}$. Below, we consider in detail the moment of inertia of the bound Bose gas.

For the bound Bose system, we consider a mesoscopic rotational state [17] with a thermal angular velocity $\mathbf{\Omega}$. The rotational version of Eq. (4.2) reads

$$d\Xi_{\mathbf{\Omega}}=-\mathcal{S}dT-\mathcal{N}d\mu-\mathbf{L}\cdot d\mathbf{\Omega}, \tag{6.2}$$

where $\mathbf{L}$ is the bound boson angular momentum. Eq. (4.11) gets replaced by

$$h_{\mathbf{\Omega}}(\mathbf{p},\mathbf{r})=\frac{p^2}{2M}+\frac{1}{2}M\mathbf{r}\cdot\hat{\omega}^2\cdot\mathbf{r}-\mathbf{\Omega}\cdot(\mathbf{r}\times\mathbf{p}), \tag{6.3}$$

so that Eq. (5.1) now reads

$$q_{\mathbf{\Omega}}(T)=\int\int\left(\frac{d^3rd^3p}{(2\pi\hbar)^3}\right)e^{-h_{\mathbf{\Omega}}(\mathbf{p},\mathbf{r})/k_BT} =[q(T)/\sqrt{\mathrm{Det}(1-\hat{\omega}^{-2}\hat{\Omega}^2)}], \tag{6.4}$$

where the matrix $\hat{\omega}^2$ is written in Eq. (4.8) and

$$\hat{\Omega}^2=\begin{pmatrix} (\Omega_2^2+\Omega_3^2) & -\Omega_1\Omega_2 & -\Omega_1\Omega_3 \\ -\Omega_1\Omega_2 & (\Omega_1^2+\Omega_3^2) & -\Omega_2\Omega_3 \\ -\Omega_1\Omega_3 & -\Omega_2\Omega_3 & (\Omega_1^2+\Omega_2^2) \end{pmatrix}. \tag{6.5}$$

From Eqs. (4.4) and (6.4), it follows that

$$\Xi_{\mathbf{\Omega}}(T,\mu)=[\Xi(T,\mu)/\sqrt{\mathrm{Det}(1-\hat{\omega}^{-2}\hat{\Omega}^2)}], \tag{6.6}$$

The fluid moment of inertia tensor has the matrix elements

$$\hat{I}_{ij}=\lim_{\mathbf{\Omega}\to\mathbf{0}}\left(\frac{\partial L_i}{\partial\Omega_j}\right)_{T,\mu}=-\lim_{\mathbf{\Omega}\to\mathbf{0}}\left(\frac{\partial^2\Xi_{\mathbf{\Omega}}}{\partial\Omega_i\partial\Omega_j}\right)_{T,\mu}. \tag{6.7}$$

Equations (4.8), (6.5), (6.6), and (6.7) imply (in the unordered phase)

$$\hat{I}=-\Xi(T,\mu)\begin{pmatrix} \left(\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}\right) & 0 & 0 \\ 0 & \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_3^2}\right) & 0 \\ 0 & 0 & \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right) \end{pmatrix} \quad (T>T_c). \tag{6.8}$$

In the unordered phase, obeying Eq. (5.2), one finds that Eq. (6.8) is precisely what would be expected from a normal fluid with geometric moment of inertia

$$\hat{I}_{ij}=\int d^3r\overline{\rho(\mathbf{r})}(r^2\delta_{ij}-r_ir_j) \quad (T>T_c), \tag{6.9}$$

where $\overline{\rho(\mathbf{r})}$ is the mean mass density of the atoms.

In the ordered phase $(T<T_c)$, the moment of inertia of the particles over and above the condensate is given by Eq. (6.8) with $\mu=0$, i.e.,

$$\hat{I}^{\mathrm{excitation}}=\zeta(4)\hbar\overline{\omega}\left(\frac{k_BT}{\hbar\overline{\omega}}\right)^4\times\begin{pmatrix} \left(\frac{1}{\omega_2^2}+\frac{1}{\omega_3^2}\right) & 0 & 0 \\ 0 & \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_3^2}\right) & 0 \\ 0 & 0 & \left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right) \end{pmatrix} \quad (T<T_c). \tag{6.10}$$

The question of superfluidity concerns the magnitude of the moment of inertia of those particles within the condensate. For $T<T_c$, we use the notation that $\hat{I}$ denotes the moment of inertia of the excited bosons, and $\hat{J}$ represents the moment of inertia of the Bose condensate. If the moment of inertia of the particles in the condensate were zero, then the condensate particles would all be "superfluid."

Let $\psi_0(\mathbf{r})$ be the normalized $[\int d^3r|\psi_0(\mathbf{r})|^2=1]$ Bose condensation state. From the geometric viewpoint, the moment of inertia of the condensate would be given by

$$J_{ij}^{\mathrm{geometric}}=\mathcal{N}_0\int d^3r|\psi_0(\mathbf{r})|^2(r^2\delta_{ij}-r_ir_j); \tag{6.11}$$

i.e.,

$$\hat{J}^{\mathrm{geometric}}=\frac{\hbar\mathcal{N}_0}{2}\begin{pmatrix} \left(\frac{1}{\omega_2}+\frac{1}{\omega_3}\right) & 0 & 0 \\ 0 & \left(\frac{1}{\omega_1}+\frac{1}{\omega_3}\right) & 0 \\ 0 & 0 & \left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right) \end{pmatrix}. \tag{6.12}$$

The *physical* Bose condensate moment of inertia tensor is in reality

$$J_{ij}^{\mathrm{physical}}=\mathcal{N}_0\sum_{\kappa}\left(\frac{\langle\psi_0|l_i|\psi_\kappa\rangle\langle\psi_\kappa|l_j|\psi_0\rangle+\langle\psi_0|l_j|\psi_\kappa\rangle\langle\psi_\kappa|l_i|\psi_0\rangle}{\epsilon_\kappa-\epsilon_0}\right), \tag{6.13}$$

where $\mathbf{l}=-i\hbar(\mathbf{r}\times\nabla)$. One may derive Eq. (6.13) by treating the rotational coupling $\Delta h=-\mathbf{\Omega}\cdot\mathbf{l}$ to second order perturbation theory in the energy $\Delta\epsilon_0(\mathbf{\Omega})$ as $\mathbf{\Omega}\to\mathbf{0}$. Equation (6.13) evaluates to

$$\hat{J}^{\mathrm{physical}}=\frac{\hbar\mathcal{N}_0}{2}\begin{pmatrix} \left(\frac{(\omega_2-\omega_3)^2}{\omega_2\omega_3(\omega_2+\omega_3)}\right) & 0 & 0 \\ 0 & \left(\frac{(\omega_1-\omega_3)^2}{\omega_1\omega_3(\omega_1+\omega_3)}\right) & 0 \\ 0 & 0 & \left(\frac{(\omega_1-\omega_2)^2}{\omega_1\omega_2(\omega_1+\omega_2)}\right) \end{pmatrix}. \tag{6.14}$$

If there exists an axis of symmetry, then the condensate moment of inertia corresponding to that axis vanishes. For an experimental example, if $\omega_1 = \omega_2 \neq \omega_3$, then the 3 axis is an axis of symmetry and $J_{33}^{\rm physical} = 0$ as implied by Eq. (6.14). In this sense, for the ideal Bose gas, the superfluid fraction is the same as the condensate fraction in Eq. (5.6),

$$\eta_{\rm superfluid}(T) = \left\{ 1 - \left(\frac{T}{T_c} \right)^3 \right\}. \tag{6.15}$$

If we were to employ an axis that is not a symmetry axis, i.e., if the angular momentum about that axis were not conserved, then even the superfluid would give some contribution to the moment of inertia.

Finally, including the scattering length will have some effect on the magnitude of the superfluid fraction. A long review of many different methods for computation for interaction effects has been given in [18]. An unusual two phase equilibrium point of view towards Bose condensation in interacting systems has been noted in [19], which has achieved a certain amount of success for the case of liquid ^{4}He.

VII. CONCLUSIONS

We have employed the heat capacity C in the relationship

$$\left(\frac{\delta T}{T} \right) \approx \sqrt{\frac{k_B}{C}} \tag{7.1}$$

in order to place lower bounds on possible system temperatures. The minimum system temperatures were estimated based on the notion that the temperature is ''well defined'' only if temperature fluctuations are small $\delta T \ll T$. The third law of thermodynamics, in the form $\lim_{(T\to 0)} C = 0$, dictates that the condition $\delta T \ll T$ is harder to achieve as the temperature is lowered.

For example, one may achieve low temperatures by adiabatic demagnetization [20]. The thermodynamics of the method is well illustrated by a system of $\mathcal{N}$ noninteracting two level particles, each with possible energies

$$E_{\pm} = \pm \Delta. \tag{7.2}$$

Such a physical system might consist of $\mathcal{N}$ nuclear one-half spins in a magnetic field. The mean energy of such a system,

$$\mathcal{E} = -\mathcal{N}\Delta \tanh(\Delta/k_B T), \tag{7.3}$$

implies a heat capacity $C = (\partial \mathcal{E}/\partial T)$ given by

$$C = \mathcal{N} k_B \left(\frac{(\Delta/k_B T)}{\cosh(\Delta/k_B T)} \right)^2. \tag{7.4}$$

The temperature fluctuation of a system of noninteracting two-level particles follows from Eqs. (7.1) and (7.4) to be

$$\left(\frac{\delta T}{T} \right) \approx \left(\frac{1}{\sqrt{\mathcal{N}}} \right) \left(\frac{\cosh(\Delta/k_B T)}{(\Delta/k_B T)} \right). \tag{7.5}$$

Thus, for a system of $\mathcal{N} \sim 10^6$ two level particles, it is only possible to achieve a temperature as low as $T_{\rm min} \sim 0.2(\Delta/k_B)$. For a system of $\mathcal{N} \sim 10^6$ trapped bosons, as shown in this work, it is possible to achieve a temperature as low as $T_{\rm min} \sim 0.05 T_c$. It is evident on the grounds of temperature fluctuations alone that the bound boson systems are the more likely example for the very lowest temperatures. Indeed, this has turned out to be true in the laboratory.

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